

# AVOIDING ZERO-SUM SUBSEQUENCES OF PRESCRIBED LENGTH OVER THE INTEGERS

C. AUGSPURGER, M. MINTER, K. SHOUKRY, P. SISSOKHO, AND K. VOSS

MATHEMATICS DEPARTMENT, ILLINOIS STATE UNIVERSITY  
NORMAL, IL 61790-4520, U.S.A.

**ABSTRACT.** Let  $t$  and  $k$  be positive integers, and let  $I_k = \{i \in \mathbb{Z} : -k \leq i \leq k\}$ . Let  $s'_t(I_k)$  be the smallest positive integer  $\ell$  such that every zero-sum sequence  $S$  over  $I_k$  of length  $|S| \geq \ell$  contains a zero-sum subsequence of length  $t$ . If no such  $\ell$  exists, then let  $s'_t(I_k) = \infty$ .

In this paper, we prove that  $s'_t(I_k)$  is finite if and only if every integer in  $[1, D(I_k)]$  divides  $t$ , where  $D(I_k) = \max\{2, 2k - 1\}$  is the Davenport constant of  $I_k$ . Moreover, we prove that if  $s'_t(I_k)$  is finite, then  $t + k(k - 1) \leq s'_t(I_k) \leq t + (2k - 2)(2k - 3)$ . We also show that  $s'_t(I_k) = t + k(k - 1)$  holds for  $k \leq 3$  and conjecture that this equality holds for any  $k \geq 1$ .

## 1. INTRODUCTION AND MAIN RESULTS

We shall follow the notation in [18], by Gryniewicz. Let  $\mathbb{N}$  be the set of positive integers. Let  $G_0$  a subset of an abelian group  $G$ . A sequence over  $G_0$  is an unordered list of terms in  $G_0$ , where repetition is allowed. The set of all sequences over  $G_0$  is denoted by  $\mathcal{F}(G_0)$ . A sequence with no term is called *trivial* or *empty*. If  $S$  is a sequence with terms  $s_i$ ,  $1 \leq i \leq n$ , we write  $S = s_1 \cdot \dots \cdot s_n = \prod_{i=1}^n s_i$ . We say that  $R$  is a *subsequence* of  $S$  if any term in  $R$  is in  $S$ . If  $R$  and  $T$  are subsequences of  $S$  such that  $S = R \cdot T$ , then  $R$  is the *complementary* sequence of  $T$  in  $S$ , and vice versa. We also write  $T = S \cdot R^{-1}$  and  $R = S \cdot T^{-1}$ . For every sequence  $S = s_1 \cdot \dots \cdot s_n$  over  $G_0$ ,

- $-S = (-s_1) \cdot \dots \cdot (-s_n)$
- the *length* of  $S$  is  $|S| = n$ ;
- the *sum* of  $S$  is  $\sigma(S) = s_1 + s_2 + \dots + s_n$ ;
- the *subsequence-sum* of  $S$  is  $\Sigma(S) = \{\sigma(R) : R \text{ is a subsequence of } S\}$ .

For any sequence  $R$  over  $G_0$  and any integer  $d \geq 0$ ,

$$R^{[0]} \text{ is the trivial sequence, and } R^{[d]} = \underbrace{R \cdot \dots \cdot R}_d \text{ for } d > 0.$$

A sequence with sum 0 is called *zero-sum*. The set of all zero-sum sequences over  $G_0$  is denoted by  $\mathcal{B}(G_0)$ . A zero-sum sequence is called *minimal* if it does not contain

---

*Key words and phrases.* zero-sum sequence over  $\mathbb{Z}$ ; no zero-sum subsequence of a given length.  
*Corresponding author:* psissok@ilstu.edu.

a proper zero-sum subsequence. The *Davenport constant* of  $G_0$ , denoted by  $D(G_0)$  is the maximum length of a minimal zero-sum sequence over  $G_0$ . The research on zero-sum theory is quite extensive when  $G$  is a finite abelian groups (e.g., see [5, 8, 10, 11] and the references therein). However, there is less activity when  $G$  is infinite (e.g., see [1, 6] and the references therein). The study of the particular case  $G = \mathbb{Z}^r$  was explicitly suggested by Baeth and Geroldinger [2] due to their relevance to direct-sum decompositions of modules. In a recent paper, Baeth et al. [3] studied the Davenport constant of  $G_0 \subseteq \mathbb{Z}^r$ . The Davenport constant of an interval in  $\mathbb{Z}$  was first derived (see Theorem 1) by Lambert [16] (also see [7, 20, 21] for related work.) In a recent paper, Plagne and Tringali [17], studied the Davenport constant of the cartesian product of intervals of  $\mathbb{Z}$ .

For any integers  $x$  and  $y$  with  $x \leq y$ , let  $[x, y] = \{i \in \mathbb{Z} : x \leq i \leq y\}$ . For  $k \in \mathbb{N}$ , let  $I_k = [-k, k]$ .

**Theorem 1** (Lambert [16]).  $D(I_k) = \max\{2, 2k - 1\}$  for any  $k \in \mathbb{N}$ .

For  $G$  finite and  $G_0 \subseteq G$ , let  $\mathbf{s}_t(G_0)$  be the smallest integer  $\ell$  such that any sequence  $S \in \mathcal{F}(G_0)$  of length  $|S| \geq \ell$  contains a zero-sum subsequence of length  $t$ . If  $t = \exp(G)$ , then  $\mathbf{s}_t(G_0)$  is called the Erdős–Ginzburg–Ziv constant and is denoted by  $\mathbf{s}(G)$ . In 1961, Erdős–Ginzburg–Ziv [8] proved that  $\mathbf{s}(\mathbb{Z}_n) = 2n - 1$ . Reiher [19] proved that  $\mathbf{s}(\mathbb{Z}_p \oplus \mathbb{Z}_p) = 4p - 3$  for any prime  $p$ . In general, if  $G$  has rank two, say  $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$  with  $1 \leq n_1 \mid n_2$ , then  $\mathbf{s}(G) = 2n_1 + 2n_2 - 3$  (see Theorem 5.8.3 in Geroldinger–Halter–Koch [12]). For groups of higher rank, we refer the reader to Fan–Gao–Zhong [9]. More recently, Gao et al [14] proved that for any integer  $k \geq 2$  and any finite  $G$  with exponent  $n = \exp(G)$ , if the difference  $n - |G|/n$  is large enough, then  $\mathbf{s}_{kn}(G) = kn + D(G) - 1$ .

Observe that if  $G$  is torsion-free and  $G_0 \subseteq G$ , then for any nonzero  $g \in G_0$  and for any  $d \in \mathbb{N}$ , the sequence  $g^{[d]} \in \mathcal{F}(G_0)$  does not contain a zero-sum subsequence. Thus, we will work with the following analogue of  $\mathbf{s}_t(G_0)$ .

**Definition 2.**<sup>1</sup> For any subset  $G_0 \subseteq G$ , let  $\mathbf{s}'_t(G_0)$  be the smallest positive integer  $\ell$  such that any sequence  $S \in \mathcal{B}(G_0)$  of length  $|S| \geq \ell$  contains a zero-sum subsequence of length  $t$ . If no such  $\ell$  exists, then let  $\mathbf{s}'_t(G_0) = \infty$ .

If  $t = \exp(G)$  is finite, then we denote  $\mathbf{s}_t(G_0)$  by  $\mathbf{s}(G)$ . Let  $r \in \mathbb{N}$  and assume that  $G \cong \mathbb{Z}_n^r$ . Then  $G$  has Property D if any sequence  $S \in \mathcal{F}(G)$  of length  $\mathbf{s}(G) - 1$  that does not admit a zero-sum subsequence has the form  $S = T^{[n-1]}$  for some  $T \in \mathcal{F}(G)$ . Zhong found the following interesting connections between  $\mathbf{s}(G)$  and  $\mathbf{s}'(G)$  (see the Appendix for their proofs).

**Lemma 3** (Zhong [22]). Let  $G$  be a finite abelian group.

- (i) If  $\gcd(\mathbf{s}(G) - 1, \exp(G)) = 1$ , then  $\mathbf{s}'(G) = \mathbf{s}(G)$ .
- (ii) If  $G \cong \mathbb{Z}_n^r$ , with  $n \geq 3$  and  $r \geq 2$ . Suppose that  $\mathbf{s}(G) = c(n - 1) + 1$  and  $G$  has Property D. If  $\gcd(\mathbf{s}(G) - 1, n) = c$ , then  $\mathbf{s}'(G) < \mathbf{s}(G)$ .

<sup>1</sup>This formulation was suggested to us by Geroldinger and Zhong [15].

**Remark 4** (Zhong [22]).

- (i) If  $G \cong \mathbb{Z}_n^2$  with  $n$  odd, then  $s'(G) = s(G)$ .
- (ii) If  $G \cong \mathbb{Z}_{2^h}^2$  with  $h \geq 2$ , then  $s'(G) = s(G) - 1$ .

In this paper, we prove the following results about  $s'_t(I_k)$ , where  $I_k = [-k, k]$ .

**Theorem 5.** *Let  $k$  and  $t$  be positive integers.*

- (i)  $s'_t(I_k)$  is finite, then every integer in  $[1, D(I_k)]$  divides  $t$ .
- (ii) If every integer in  $[1, D(I_k)]$  divides  $t$ , then

$$t + k(k-1) \leq s'_t(I_k) \leq t + (2k-2)(2k-3).$$

**Corollary 1.** *Let  $t \in \mathbb{N}$  and  $k \in [1, 3]$ . Then  $s'_t(I_k) = t + k(k-1)$  if and only if every integer in  $[1, D(I_k)]$  divides  $t$ .*

**Conjecture 6.** *Corollary 1 holds for any  $k \in \mathbb{N}$ .*

## 2. PROOFS OF THE MAIN RESULTS

For any integers  $a$  and  $b$ , we denote  $\gcd(a, b)$  by  $(a, b)$ . We use the abbreviations z.s.s and z.s.sb for *zero-sum sequence(s)* and *zero-sum subsequence(s)*, respectively. The letters  $k$  and  $t$  will denote positive integers throughout the paper.

The following lemma gives a lower bound for  $s'_t(I_k)$ .

**Lemma 7.** *Consider the z.s.s  $U = k \cdot (-1)^{[k]}$  and  $V = (k-1) \cdot (-1)^{[k-1]}$ . Then,  $S = U^{[\frac{t}{k+1}-1]} \cdot V^{[k]}$  and  $R = U^{[k-1]} \cdot V^{[\frac{t}{k}-1]}$  are z.s.s that do not contain a z.s.sb of length  $t$ . Thus,  $s'_t(I_k) \geq t + k(k-1)$ .*

*Proof.* We prove the lemma for  $S$  only since the proof for  $R$  is similar. By contradiction, assume that  $S$  contains a z.s.sb of length  $t$ . Since  $\sigma(S) = 0$ , it follows that  $S$  also contains a z.s.sb  $S'$  of length  $|S| - t = k(k-1) - 1$ . Moreover,  $S'$  can be written as  $S' = k^{[a]} \cdot (k-1)^{[b]} \cdot (-1)^{[c]}$  for some nonnegative integers  $a$ ,  $b$ , and  $c$ . Hence  $\sigma(S') = ak + b(k-1) - c = 0$  and  $a + b + c = |S'| = k^2 - k - 1$ . Thus

$$(a+1)(k+1) = k(k-b).$$

Since  $a, b, k \geq 0$ , we have  $0 < k-b \leq k$ . Since  $(k, k+1) = 1$ , we obtain that  $k+1$  divides  $k-b$ , which is a contradiction. Thus  $s'_t(I_k) \geq |S| + 1 = t + k(k-1)$ .  $\square$

**Example 8.** *For  $k = 3$ ,  $S = (3 \cdot -1 \cdot -1 \cdot -1)^{[14]} \cdot (2 \cdot -1 \cdot -1)^{[3]}$  is a z.s.s of length 65 over  $[-3, 3]$  which does not contain a z.s.sb of length  $t = 60$ .*

**Lemma 9.** *Let  $a, b, x \in \mathbb{N}$ . If  $S = a^{[\frac{b}{(a,b)}} \cdot (-b)^{[\frac{a}{(a,b)}}]$  is a z.s.s, then the length of any z.s.sb of  $S^{[x]}$  is a multiple of  $|S|$ .*

*Proof.* Let  $S'$  be a z.s.sb of  $S^{[x]}$ . Since the terms of  $S$  are  $a$  and  $-b$ , there exist nonnegative integers  $h$  and  $r$  such that  $S' = a^h \cdot (-b)^r$  and

$$(1) \quad \sigma(S') = ha - rb = 0 \Rightarrow h \frac{a}{(a,b)} = r \frac{b}{(a,b)}.$$

Since  $\left(\frac{b}{(a,b)}, \frac{b}{(a,b)}\right) = 1$ , we obtain  $\frac{b}{(a,b)}$  divides  $h$  and  $\frac{a}{(a,b)}$  divides  $r$ . Thus,  $h = p \frac{b}{(a,b)}$  and  $r = q \frac{a}{(a,b)}$  for some integers  $p$  and  $q$ . Substituting  $h$  and  $r$  back into (1) yields  $p = q$ . Thus,

$$|S'| = h + r = p \frac{b}{(a,b)} + q \frac{a}{(a,b)} = p|S|.$$

□

**Lemma 10.** *If  $\mathcal{S}'_t(I_k)$  is finite, then every odd integer in  $[1, D(I_k)]$  divides  $t$ .*

*Proof.* Since the lemma is trivial for  $k = 1$ , we assume that  $k \geq 2$ . Then  $D(I_k) = 2k - 1$  by Theorem 1. Let  $\ell = 2c - 1$  be an odd integer in  $[3, D(I_k)]$ , and consider the minimal z.s.s  $S = c^{[c-1]} \cdot (-c + 1)^{[c]}$ . Then, for any  $x \in \mathbb{N}$ , it follows from Lemma 9 that for any z.s.sb  $R$  of  $S^{[x]}$ ,  $|R|$  divides  $|S| = 2c - 1 = \ell$ . Since  $\ell$  does not divide  $t$ , there is no z.s.sb of  $S^{[x]}$  whose length is equal to  $t$ . Since  $x$  is arbitrary, it follows that  $\mathcal{S}'_t(I_k)$  can be arbitrarily large. This proves the lemma. □

To prove the upper bound in Theorem 5(ii), we will use the following lemma which is a directly application a well-known fact: “Any sequence of  $n$  integers contains a nonempty subsequence whose sum is divisible by  $n$ ”.

**Lemma 11.** *Let  $\beta \in \mathbb{N}$  and  $X \in \mathcal{F}(\mathbb{Z})$ . If  $|X| \geq \beta$ , then there exists a factorization  $X = X_0 \cdot X_1 \cdot \dots \cdot X_r$  such that*

- (i)  $|X_0| \leq \beta - 1$  and no subsequence of  $X_0$  has a sum that is divisible by  $\beta$ .
- (ii)  $|X_j| \leq \beta$  and  $\sigma(X_j)$  is divisible by  $\beta$  for any  $j \in [1, r]$ .

We will also use the following lemmas.

**Lemma 12.** *Assume that  $k \geq 2$  and that every integer in  $[1, D(I_k)]$  divides  $t$ . Let  $S$  be a z.s.s over  $I_k = [-k, k]$  that does not contain a z.s.sb of length  $t$ . Let  $S = S_1 \cdot \dots \cdot S_h$  be a factorization of  $S$  into minimal z.s.sb  $S_i$ ,  $1 \leq i \leq h$ . If  $|S| \geq t + k(k - 1)$ , then there exists some length  $\beta$  such that  $n_\beta = |\{S_i : |S_i| = \beta, 1 \leq i \leq h\}|$  satisfies:*

$$n_\beta > (2k - 2)(2k - 3).$$

*Proof.* Recall that  $(a, b)$  denotes  $\gcd(a, b)$ . It is easy to see that

$$(2) \quad (2k - 3, 2k - 2) = (2k - 2, 2k - 1) = (2k - 3, 2k - 1) = 1.$$

Since  $k > 1$  and every integer in  $[1, D(I_k)] = [1, 2k - 1]$  is a factor of  $t$ , it follows from (2) that  $t = p(2k - 1)(2k - 2)(2k - 3)$ , for some  $p \in \mathbb{N}$ . By definition, we have

$\max_{1 \leq i \leq h} |S_i| \leq D(I_k) = 2k - 1$ . Thus, it follows from the pigeonhole principle that there exists some length  $\beta$  such that

$$n_\beta \geq \frac{t + k(k-1)}{\max_{1 \leq i \leq h} |S_i|} \geq \frac{t + k(k-1)}{2k-1} > p(2k-2)(2k-3).$$

□

**Lemma 13.** *Assume that  $k \geq 2$  and that every integer in  $[1, D(I_k)]$  divides  $t$ . Let  $S$  be a z.s.s over  $I_k = [-k, k]$  of length  $|S| \geq t + k(k-1)$  such that  $S$  does not contain a z.s.s of length  $t$ . Let  $S = S_1 \cdot \dots \cdot S_h$  be a factorization of  $S$  into minimal z.s.s  $S_i$ ,  $1 \leq i \leq h$ . Let  $L = \{|S_i| : 1 \leq i \leq h\}$ ,  $\alpha = \max_{\ell \in L} \ell$ , and let  $n_\ell = |\{S_i : |S_i| = \ell, 1 \leq i \leq h\}|$ .*

*If there exists  $\beta \in L$  such that  $n_\beta \geq \alpha - 1$ , then*

$$|S| \leq t - \beta + (\beta - 1) \max_{\ell \in L \setminus \{\beta\}} \ell.$$

**Remark 14.** *By Lemma 12, there exists  $\beta \in L$  such that  $n_\beta > (2k-2)(2k-3)$ . Moreover,  $\alpha = \max_{\ell \in L} \ell \leq D(I_k) \leq (2k-2)(2k-3) + 1$  for  $k \geq 2$ . Thus,  $n_\beta \geq \alpha$ , i.e., the hypothesis of Lemma 13 always holds.*

*Proof.* By hypothesis, there exists  $\beta \in L$  such that  $n_\beta \geq \alpha - 1$ . Given a factorization  $S = S_1 \cdot \dots \cdot S_h$  into minimal z.s.s  $S_i$ ,  $1 \leq i \leq h$ , consider the sequence of lengths in  $L \setminus \{\beta\}$ :

$$X = \prod_{i=1, |S_i| \neq \beta}^h |S_i| = \prod_{\ell \in L \setminus \{\beta\}} \ell^{[n_\ell]}.$$

It follows from Lemma 11 that there exists a factorization  $X = X_0 \cdot X_1 \dots X_r$  such that

$$(3) \quad |X_0| \leq \beta - 1, \text{ and no subsequence } X_0 \text{ has a sum that is divisible by } \beta.$$

$$(4) \quad |X_j| \leq \beta \text{ and } \beta \text{ divides } \sigma(X_j) \text{ for all } j \in [1, r].$$

Thus,

$$(5) \quad \sigma(X_j) = \sum_{x \in X_j} x \leq |X_j| \cdot \max_{x \in X_j} x \leq \beta \alpha \text{ for all } j \in [1, r].$$

Note that (4), (5), and the hypothesis on  $\beta$  imply that:

$$\beta \text{ divides } t; n_\beta \geq \alpha - 1; \sigma(X_j) \leq \alpha \beta; \text{ and } \beta \text{ divides } \sigma(X_j) \text{ for all } j \in [1, r].$$

Thus, if

$$\beta n_\beta + \sum_{j=1}^r \sigma(X_j) \geq t,$$

then there exists a nonnegative integer  $n'_\beta \leq n_\beta$  and a subset  $Q \subseteq [1, r]$  such that

$$\beta n'_\beta + \sum_{q \in Q} \sigma(X_q) = t.$$

Then  $S$  would contain a z.s.s of length  $t$  obtained by concatenating  $n'_\beta$  minimal z.s.sb of  $S$  of length  $\beta$  and all the z.s.s of  $S$  whose lengths are in  $X_q$  for all  $q \in Q$ . This contradicts the hypothesis of the theorem. Thus,  $\beta n_\beta + \sum_{j=1}^r \sigma(X_j) < t$  must hold. Since  $\beta$  divides both  $t$  and  $\sum_{j=1}^r \sigma(X_j)$ , we obtain

$$\beta n_\beta + \sum_{j=1}^r \sigma(X_j) \leq t - \beta.$$

Thus, it follows from the definition of  $X$  and  $X_j$ ,  $0 \leq j \leq r$ , that

$$\begin{aligned} |S| &= \sum_{\ell \in L} \ell n_\ell = \beta n_\beta + \sigma(X) \\ &= \beta n_\beta + \sum_{j=1}^r \sigma(X_j) + \sigma(X_0) \\ (6) \quad &\leq t - \beta + \sigma(X_0). \end{aligned}$$

Next, it follows from (3) and (6) that

$$|S| \leq t - \beta + \sigma(X_0) \leq t - \beta + |X_0| \max_{\ell \in L \setminus \{\beta\}} \ell \leq t - \beta + (\beta - 1) \max_{\ell \in L \setminus \{\beta\}} \ell.$$

□

*Proof of Theorem 5.* We first prove part (i). Suppose that  $\mathbf{s}'_t(I_k)$  is finite. Then it follows from Lemma 10 that every odd integer in  $[1, D(I_k)]$  divides  $t$ . Thus, it remains to show that if  $a$  is an even integer in  $[1, D(I_k)]$ , then  $a$  divides  $t$ .

**Case 1:**  $a = 2^e$  for some integer  $e \geq 1$ .

Lemma 9 implies that for any  $p \in \mathbb{N}$ , the sequence  $S = (1 \cdot -1)^{[p]}$  is a z.s.s whose z.s.sb have lengths that are multiples of 2. Therefore, if 2 does not divide  $t$ , then  $\mathbf{s}'_t(I_k) \geq |S| = 2p$ , where  $p$  can be chosen to be arbitrarily large. Thus, 2 divides  $t$  if  $\mathbf{s}'_t(I_k)$  is finite.

Now assume that  $e > 1$ . Since the gcd of two numbers divides their difference,  $(a/2 - 1, a/2 + 1) \leq 2$ . But 2 does not divide  $a/2 - 1$  or  $a/2 + 1$ ; and so  $(a/2 - 1, a/2 + 1) = 1$ . Lemma 10 implies that for any  $p \in \mathbb{N}$ , the sequence  $S^{[p]}$  with  $S = (a/2 - 1)^{[a/2+1]} \cdot (-a/2 - 1)^{[a/2-1]}$  is a z.s.s whose z.s.sb have lengths that are multiples of  $|S| = (a/2 + 1) + (a/2 - 1) = a$ . Thus, if  $a$  does not divide  $t$ , we can obtain arbitrarily long z.s.s over  $I_k = [-k, k]$  that do not contain z.s.sb of length  $t$ , because  $p$  can be chosen to be arbitrarily large. Thus,  $a$  divides  $t$  if  $\mathbf{s}'_t(I_k)$  is finite.

**Case 2:**  $a$  is not a power of 2.

Then  $a = 2^e j$ , where  $e$  and  $j$  are nonnegative integers such that  $j$  is odd. By Lemma 10,  $j$  divides  $t$ , and it follows from Case 1 that  $2^e$  divides  $t$ . Since  $j$  is odd,  $(2^e, j) = 1$ . Since  $2^e$  and  $j$  are factors of  $t$ , it follows that  $2^e j$  divides  $t$ .

Thus, it follows from Case 1, Case 2, and Lemma 10 that every integer in  $[1, D(I_k)]$  divides  $t$ .

Since the lower bound of  $\mathbf{s}'_t(I_t)$  in Theorem 5(ii) follows from Lemma 7, it remains to prove its upper bound. Let  $k, t \in \mathbb{N}$  be such that every integer in  $[1, D(I_k)]$  divides  $t$ . In particular,  $t$  is even. Let  $S$  be an arbitrary z.s.s over  $I_k = [-k, k]$  that does not contain a z.s.sb of length  $t$ .

If  $k = 1$ , then it follows from Theorem 1 that  $D(I_k) = 2$ . Thus, 2 divides  $t$  and  $|S| = x_1 + 2x_2$  for some nonnegative integers  $x_1$  and  $x_2$ . If  $|S| \geq t$ , then  $x_1 \geq 2$  or  $x_2 \geq t/2$  (because  $t$  is even). This implies that there exist nonnegative integers  $x'_1 \leq x_1$  and  $x'_2 \leq x_2$  such that  $x'_1 + 2x'_2 = t$ . Thus  $S' = (1 \cdot -1)^{[x'_2]} \cdot 0^{[x'_1]}$  is a z.s.sb of  $S$  of length  $t$ , which contradicts the fact that  $S$  does not contain a z.s.s of length  $t$ . Hence  $|S| \leq t - 1$ , and  $\mathbf{s}'_t(I_k) \leq |S| + 1 = t$ .

Now assume  $k \geq 2$ . Since  $S$  was arbitrarily chosen, it follows that if  $|S| \leq t + k(k - 1) - 1$ , then

$$\mathbf{s}'_t(I_k) \leq |S| + 1 \leq t + k(k - 1) \leq t + (2k - 2)(2k - 3),$$

and the upper bound in Theorem 5(ii) follows. So we may assume that  $|S| \geq t + k(k - 1)$ . Let  $S = S_1 \cdots S_h$  be a factorization of  $S$  into minimal z.s.sb. Let  $L = \{|S_i| : 1 \leq i \leq h\}$ ,  $\alpha = \max_{\ell \in L} \ell$ , and let  $n_\ell = |\{S_i : |S_i| = \ell, 1 \leq i \leq h\}|$ . Then Remark 14 implies that there exists  $\beta \in L$  is such that  $n_\beta \geq \alpha - 1$ . If  $\beta = \alpha$ , then Lemma 13 yields

$$|S| \leq t - \alpha + (\alpha - 1) \max_{\ell \in L \setminus \{\alpha\}} \ell \leq t - \alpha + (\alpha - 1)^2.$$

If  $1 \leq \beta \leq \alpha - 1$ , then Lemma 13 also yields

$$\begin{aligned} |S| &\leq t + \max_{1 \leq \beta \leq \alpha - 1} \left( -\beta + (\beta - 1) \max_{\ell \in L \setminus \{\beta\}} \ell \right) \\ &\leq t + \max_{1 \leq \beta \leq \alpha - 1} (-\beta + (\beta - 1)\alpha) \\ &= t + (-(\alpha - 1) + (\alpha - 2)\alpha) \\ &= t - \alpha + (\alpha - 1)^2. \end{aligned}$$

So in all cases, we obtain

$$(7) \quad |S| \leq t - \alpha + (\alpha - 1)^2 \leq t - (2k - 1) + (2k - 2)^2,$$

where we used the fact  $\alpha \leq D(I_k) = 2k - 1$ . Since  $S$  was chosen to be an arbitrary z.s.s over  $I_k = [-k, k]$  which does not contain a z.s.sb of length  $t$ , it follows that

$$\mathbf{s}'_t(I_k) \leq |S| + 1 \leq t - (2k - 1) + (2k - 2)^2 + 1 = t + (2k - 2)(2k - 3).$$

□

*Proof of Corollary 1.* For  $k \in \{1, 2\}$ , the corollary holds since the upper and lower bounds of  $\mathbf{s}'_t(I_k)$  given by Theorem 5 are both equal to  $t + k(k - 1)$ .

For  $k = 3$ , it also follows from Theorem 5 that  $t + 6 \leq \mathbf{s}'_t(I_3) \leq t + 12$ . Thus, it remains to show that if  $S$  is an arbitrary z.s.s over  $I_3$  which does not contain a z.s.s of length  $t$ , then  $|S| \neq t + d$  for all  $d \in [6, 11]$ .



Consider a factorization  $S = S_1 \cdot \dots \cdot S_h$  into minimal z.s.s  $S_i$ ,  $i \in [1, h]$ . Let  $L = \{|S_i| : 1 \leq i \leq h\}$ ,  $\alpha = \max_{\ell \in L} \ell$ , and let  $n_\ell = |\{S_i : |S_i| = \ell, 1 \leq i \leq h\}|$ . Thus,  $\alpha \leq D(I_3) = 5$ . If  $\alpha \leq 4$ , then Lemma 13 yields

$$|S| \leq t + \max_{1 \leq \alpha \leq 4} ((\alpha - 1)^2 - \alpha) = t + (4 - 1)^2 - 4 = t + 5.$$

Thus, we may assume that  $\alpha = \max_{\ell \in L} \ell = 5$  for any factorization of  $S$ .

If  $\beta \in \{1, 2\}$  and  $n_\beta \geq 4$ , then Lemma 13 yields

$$|S| \leq t + \max_{\beta \in \{1, 2\}} ((\beta - 1)\alpha - \beta) = t + (2 - 1)5 - 2 = t + 3.$$

Next, suppose that  $R$  is a z.s.sb of  $S$  with length at least 4. Then  $R \cdot -R$  can be trivially factorize into  $|W| \geq 4$  z.s.s of length 2. This would yields a new factorization  $S = S'_1 \cdot \dots \cdot S'_h$  with  $n_2 \geq 4 \geq n_5 - 1$ , which would imply that  $|S| < t + 5$  by the above analysis.

Also note that if  $n_\ell \geq t/\ell$  holds for some length  $\ell \in L$ , then we obtain a z.s.sb of  $S$  of length  $t$  by concatenating  $t/\ell$  z.s.sb of length  $\ell$  in  $S$ . This would contradict the definition of  $S$ . Thus, we can assume that  $n_\ell \leq t/\ell - 1$  for all  $\ell \in L \subseteq [1, 5]$ .

To recapitulate, we may assume that for any factorization  $S = S_1 \cdot \dots \cdot S_h$ , with  $S_L = \prod_{i=1}^h |S_i|$  and  $n_\ell = |\{S_i : |S_i| = \ell, 1 \leq i \leq h\}|$ , we have:

- (i)  $S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 3^{[n_3]} \cdot 2^{[n_2]} \cdot 1^{[n_1]}$ , where  $0 \leq n_\ell \leq t/\ell - 1$  for  $\ell \in [1, 5]$ ;  $n_5 \geq 1$ ; and  $n_1, n_2 \leq 3$ .
- (ii) There is a one-to-one correspondence between the subsequences  $S'_L$  of  $S_L$  and the z.s.s  $S'$  of  $S$  with length  $\sigma(S'_L)$ .
- (iii) If  $R$  is z.s.s over  $I_3$  such that  $|R| \geq 4$ , then  $R$  and  $-R$  cannot both be subsequences of  $S$ .
- (iv) If  $R$  is a minimal z.s.sb of  $S$  such that  $|R| = 5$ , then  $R = 3^{[2]} \cdot (-2)^{[3]}$ .  
(This follows from (iii) and the fact  $A = 3^{[2]} \cdot (-2)^{[3]}$  and  $-A$  are the only minimal z.s.s of length 5 over  $I_3 = [-3, 3]$ . Thus, if  $-A$  is the z.s.sb of  $S$ , then we can analyze  $-S$  instead of  $S$ .)

We now prove the following claims.

**Claim 1:** *If  $5 \cdot 3^{[4]}$  is a subsequence of  $S_L$ , then  $|S| \neq t + d$  for all  $d \in [6, 11]$*

If  $n_4 + n_2 + n_1 \geq 1$ , then either  $5 \cdot 4 \cdot 3^{[4]}$ , or  $5 \cdot 3^{[4]} \cdot 2$ , or  $5 \cdot 3^{[4]} \cdot 1$  is a subsequence of  $S_L$ , which implies that  $\Sigma(S_L)$  contains all the integers in  $[6, 11]$ . Thus,  $n_4 = n_2 = n_1 = 0$ , which implies that  $S_L = 5^{[n_5]} \cdot 3^{[n_3]}$ . If  $n_5 \leq 1$ , then

$$|S| = \sigma(S_L) = 5n_5 + 3n_3 \leq 5 + 3(t/3 - 1) < t + 5.$$

Thus, we may assume that

$$S_L = 5^{[n_5]} \cdot 3^{[n_3]}, \text{ where } n_5 \geq 2 \text{ and } n_3 \geq 4.$$

Then  $\Sigma(S_L)$  contain all the integers in  $[6, 11] \setminus \{7\}$ ; and so  $|S| \neq t + d$  for  $d \in [6, 11] \setminus \{7\}$ . It remains to show that  $|S| \neq t + 7$ .



Note that the only minimal z.s.s of length 3 over  $[-3, 3]$  are (up to sign)  $B_1 = 2 \cdot (-1)^{[2]}$  and  $B_2 = 3 \cdot -2 \cdot -1$ . Since  $5 \cdot 3^{[4]}$  is a subsequence of  $S_L$ , it follows from the assumptions (i)–(iv) (see above) that  $S' = A \cdot X \cdot Y \cdot Z \cdot W$  is a subsequence of  $S$ , where  $A = 3^{[2]} \cdot (-2)^{[3]}$  and  $X, Y, Z, W \in \{-B_1, B_1, -B_2, B_2\}$ . By inspecting the sequence  $S'$  for all possible choices of  $X, Y, Z$ , and  $W$ ; we see that  $S'$  admits a z.s.s of length 7. For instance, if  $X = Y = Z = B_2$ , then

$$S' = A \cdot B_2^{[3]} \cdot W = A^{[2]} \cdot 3 \cdot (-1)^{[3]} \cdot W$$

contains the subsequence  $3 \cdot (-1)^{[3]} \cdot W$ , which is a z.s.s of length  $4 + |W| = 7$ . Hence,  $|S| \neq t + 7$ . Thus,  $|S| \neq t + d$  for all  $d \in [6, 11]$ .

**Claim 2:** *If  $5 \cdot 4^{[2]} \cdot 3$  is a subsequence of  $S_L$ , then  $|S| \neq t + d$  for all  $d \in [6, 11]$ .*

If  $n_3 \geq 2$ , or  $n_2 \geq 1$ , or  $n_1 \geq 1$ , then either  $5 \cdot 4^{[2]} \cdot 3^{[2]}$ , or  $5 \cdot 4^{[2]} \cdot 3 \cdot 2$ , or  $5 \cdot 4^{[2]} \cdot 3 \cdot 1$  is a subsequence of  $S_L$ , which implies that  $\Sigma(S_L)$  contains all the integers in  $[6, 11]$ . In these cases,  $|S| \neq t + d$  for  $d \in [6, 11]$ , we are done. Thus, we may assume that  $n_2 = n_1 = 0$  and  $n_3 = 1$ , which implies that  $S_L = 5^{[n_5]} \cdot 4^{[n_3]} \cdot 3$ . If  $n_5 \leq 1$ , then

$$|S| = \sigma(S_L) = 5n_5 + 4n_4 + 3 \leq 5 + 4(t/4 - 1) + 3 < t + 5.$$

Thus, we may assume that

$$S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 3, \text{ where } n_5 \geq 2 \text{ and } n_4 \geq 2.$$

Thus,  $5^{[2]} \cdot 4^{[2]} \cdot 3$  is a subsequence of  $S_L$ , which implies that  $\Sigma(S_L)$  contain all the integers in  $[7, 11]$ . Thus  $|S| \neq t + d$  for  $d \in [7, 11]$ . It remains to show that  $|S| \neq t + 6$ .

Note that the only minimal z.s.s of length 4 over  $[-3, 3]$  are (up to sign)  $C_1 = 3 \cdot (-1)^{[3]}$  and  $C_2 = 3 \cdot 1 \cdot (-2)^{[2]}$ . Since  $5 \cdot 4^{[2]} \cdot 3$  is a subsequence of  $S_L$ , it follows from the assumptions (i)–(iv) that  $S' = A \cdot X \cdot Y \cdot Z$  is a subsequence of  $S$ , where  $A = 3^{[2]} \cdot (-2)^{[3]}$ ,  $X, Y \in \{-C_1, C_1, -C_2, C_2\}$ , and  $Z \in \{-B_1, B_1, -B_2, B_2\}$ . By inspecting the sequence  $S'$  for all possible choices of  $X, Y$ , and  $Z$ ; we see that  $S'$  admits a z.s.s of length 6. For instance, if  $X = C_1$  and  $Y = C_2$ , then

$$S' = A \cdot C_1 \cdot C_2 \cdot Z = A \cdot (3 \cdot -1 \cdot -2)^{[2]} \cdot (1 \cdot -1) \cdot Z$$

contains the subsequence  $(3 \cdot -1 \cdot -2) \cdot Z$ , which is a z.s.s of length  $3 + |Z| = 6$ . Hence,  $|S| \neq t + 6$ . Thus,  $|S| \neq t + d$  for all  $d \in [6, 11]$ .

**Claim 3:** *If  $5 \cdot 4^{[3]}$  is a subsequence of  $S_L$ , then  $|S| \neq t + d$  for all  $d \in [6, 11]$ .*

If  $n_3 \geq 1$ , then  $5 \cdot 4^{[2]} \cdot 3$  is a subsequence of  $S_L$ , and we are back in Case 2. Thus, we may assume that  $n_3 = 0$ . If  $n_2 \geq 1$  or  $n_1 \geq 2$ , then either  $5 \cdot 4^{[3]} \cdot 2$  or  $5 \cdot 4^{[3]} \cdot 1^{[2]}$  is a subsequence of  $S_L$ , which implies that  $\Sigma(S_L)$  contains all the integers in  $[6, 11]$ . Thus,  $S$  contains z.s.s of length  $\ell$  for all  $\ell \in [6, 11]$ . Hence,  $|S| \neq t + d$  for  $d \in [6, 11]$ . Thus, we may assume that  $n_2 = 0$  and  $n_1 \leq 1$ . Thus,  $S_L = 5^{[n_5]} \cdot 4^{[n_3]} \cdot 1^{[n_1]}$ . Moreover, if  $n_5 \leq 1$ , then

$$|S| = \sigma(S_L) = 5n_5 + 4n_4 + n_1 \leq 5 + 4(t/4 - 1) + 1 < t + 5.$$

Thus, we may assume that

$S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 1^{[n_1]}$ , where  $n_5 \geq 2$ ,  $n_4 \geq 3$ , and  $n_1 \leq 1$ .

Since  $5^{[2]} \cdot 4^{[3]}$  is a subsequence of  $S_L$ , it follows that  $\Sigma(S_L)$  contain all the integers in  $[8, 10]$ . Thus,  $S$  admits z.s.s of length  $\ell$  for all  $\ell \in [8, 10]$ . Hence,  $|S| \neq t + d$  for all  $d \in [8, 10]$ . Moreover, it follows from the assumptions (i)–(iv) that  $S' = A \cdot X \cdot Y \cdot Z$  is a subsequence of  $S$ , where  $A = 3^{[2]} \cdot (-2)^{[3]}$  and  $X, Y, Z \in \{-C_1, C_1, -C_2, C_2\}$ . By inspecting the sequence  $S'$  for all possible choices of  $X, Y$ , and  $Z$ ; we see that  $S'$  admits a z.s.s of length 7. Hence,  $|S| \neq t + 7$ . Overall, we obtain  $|S| \neq t + d$  for any  $d \in [7, 10]$ .

If  $5 \cdot 4^{[4]}$  is a subsequence of  $S_L$ , it again follows from the assumptions (i)–(iv) that  $S' = A \cdot X \cdot Y \cdot Z \cdot W$  is a subsequence of  $S$ , where  $A = 3^{[2]} \cdot (-2)^{[3]}$  and  $X, Y, Z, W \in \{-C_1, C_1, -C_2, C_2\}$ . By inspecting the sequence  $S'$  for all possible choices of  $X, Y, Z$ , and  $W$ ; we see that  $S'$  admits z.s.s of lengths 6 and 11. In this case,  $|S| \neq t + d$  for all  $d \in [6, 11]$ . Thus, we may assume that

$$S_L = 5^{[n_5]} \cdot 4^{[3]} \cdot 1^{[n_1]}, \text{ where } n_5 \geq 2 \text{ and } n_1 \leq 1.$$

Now, it remains to show that  $|S| \neq t + a$  for  $a \in \{6, 11\}$ . However, if  $|S| = t + a$ , then

$$5n_5 + 4(3) + n_1 = \sigma(S_L) = |S| = t + a \Rightarrow 5n_5 = t + a - 12 - n_1.$$

This is a contradiction since 5 divides  $t$  (by hypothesis) and 5 does not divide  $a - 12 - n_1$  for  $a \in \{6, 11\}$  and  $n_1 \in \{0, 1\}$ . Thus,  $|S| \neq t + d$  for all  $d \in [6, 11]$ .

Based on Claim 1–Claim 3, we may assume the following:

- (v)  $S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 3^{[n_3]} \cdot 2^{[n_2]} \cdot 1^{[n_1]}$ , where  $0 \leq n_\ell \leq t/\ell - 1$  for all  $\ell \in [1, 5]$ ;  $n_1, n_2, n_3 \leq 3$ ;  $n_4 \leq 2$ ;  $(n_4, n_3) \neq (2, 1)$ ; and  $n_5 \geq 1$ .

We will use this assumption in the following cases.

**Case 1:**  $|S| \neq t + 6$ .

Assume that  $|S| = t + 6$ . If  $n_1 \geq 1$ , then  $5 \cdot 1$  is a subsequence of  $S_L$ , which implies that  $S$  contains a z.s.sb of length  $5 + 1 = 6$  whose complementary sequence in  $S$  is a z.s.sb of length  $t$ . Thus,  $n_1 = 0$ . By a similar reasoning, we infer that the following conditions hold:  $n_3 \leq 1$ ; and  $n_4 \geq 1 \Rightarrow n_2 = 0$ . Moreover, it follows from condition (v) that  $(n_4, n_3) \neq (2, 1)$ . Consequently, either  $S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 3^{[n_3]}$  with  $n_3 \leq 1$ ,  $n_4 \leq 2$ , and  $(n_4, n_3) \neq (2, 1)$ ; or  $S_L = 5^{[n_5]} \cdot 3^{[n_3]} \cdot 2^{[n_2]}$  with  $n_3 \leq 1$  and  $n_2 \leq 2$ . Thus,

$$|S| = \sigma(S_L) \leq 5n_5 + 8 \leq 5(t/5 - 1) + 8 < t + 6,$$

which is a contradiction. Thus,  $|S| \neq t + 6$ .

**Case 2:**  $|S| \neq t + 7$ .

Assume that  $|S| = t + 7$ . If  $n_2 \geq 1$ , then  $5 \cdot 2$  is a subsequence of  $S_L$ , which implies that  $S$  contains a z.s.sb of length  $5 + 2 = 7$  whose complementary sequence in  $S$  is a z.s.sb of length  $t$ . Thus,  $n_2 = 0$ . By a similar reasoning, we infer that the following conditions hold:  $n_1 \leq 1$ ;  $n_4 \geq 1 \Rightarrow n_3 = 0$ ;  $n_3 \geq 1 \Rightarrow n_4 = 0$ ; and  $n_3 \geq 2 \Rightarrow n_1 = 0$ .

Consequently, either  $S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 1^{[n_1]}$  with  $n_4 \leq 2$  and  $n_1 \leq 1$ ; or  $S_L = 5^{[n_5]} \cdot 3 \cdot 1$ , or  $S_L = 5^{[n_5]} \cdot 3^{[n_3]}$  with  $n_3 \leq 3$ . Thus,

$$|S| = \sigma(S_L) \leq 5n_5 + 9 \leq 5(t/5 - 1) + 9 < t + 7,$$

which is a contradiction. Thus  $|S| \neq t + 7$ .

**Case 3:**  $|S| \neq t + 8$ .

Assume that  $|S| = t + 8$ . If  $n_3 \geq 1$ , then  $5 \cdot 3$  is a subsequence of  $S_L$ , which implies that  $S$  contains a z.s.sb of length  $5 + 3 = 8$  whose complementary sequence in  $S$  is a z.s.sb of length  $t$ . Thus,  $n_3 = 0$ . By a similar reasoning, we infer that  $n_4 \leq 1$ ;  $n_2 \leq 3$ ;  $n_1 \leq 2$ ;  $n_2 \geq 1 \Rightarrow n_1 = 0$ ; and  $n_1 \geq 1 \Rightarrow n_2 = 0$ . Consequently either  $S_L = 5^{[n_5]} \cdot 4 \cdot 2$ , or  $S_L = 5^{[n_5]} \cdot 4 \cdot 1^{[n_1]}$ , or  $S_L = 5^{[n_5]} \cdot 2^{[n_2]}$ , or  $S_L = 5^{[n_5]} \cdot 1^{[n_1]}$ , where  $n_2 \leq 3$  and  $n_1 \leq 2$ . Thus,

$$|S| = \sigma(S_L) \leq 5n_5 + 6 \leq 5(t/5 - 1) + 6 < t + 8,$$

which is a contradiction. Thus,  $|S| \neq t + 8$ .

**Case 4:**  $|S| \neq t + d$  for  $d \in [9, 11]$ .

Assume that  $|S| = t + 9$ . If  $n_3 \geq 1$ , then  $3$  is a subsequence of  $S_L$  which implies that  $S$  contain a z.s.s  $T$  of length  $3$ . Thus  $S' = S \cdot T^{-1}$  is a z.s.s of length  $|S| - 3 = t + 6$  which does not contain a z.s.sb of length  $6$  and, equivalently, length  $t$ . This contradicts Case 1, where we showed that no such z.s.s exists. Thus,  $n_3 = 0$ . Similarly,  $n_2 = 0$  (by Case 2) and  $n_1 = 0$  (by Case 3). Consequently,  $S_L = 5^{[n_5]} \cdot 4^{[n_4]}$  with  $n_4 \leq 2$ . Thus,

$$|S| = \sigma(S_L) = 5n_5 + 4n_4 \leq 5(t/5 - 1) + 4(2) < t + 9,$$

which is a contradiction. Thus,  $|S| \neq t + 9$ .

Since  $n_5 \geq 2$ ,  $S$  contains a z.s.s of length  $\sigma(5^{[2]}) = 10$ . Thus,  $|S| \neq t + 10$ .

Since  $n_5 \geq 1$ ,  $S$  contains a z.s.s  $T$  of length  $5$ . Thus  $S' = S \cdot T^{-1}$  is a z.s.s of length  $|S| - 5 = t + 6$  which does not contain a z.s.sb of length  $6$  and, equivalently, length  $t$ . This contradicts Case 1. Thus,  $|S| \neq t + 11$ .

In conclusion, we have shown that if  $S$  is an arbitrary z.s.s over  $I_3 = [-3, 3]$  which does not contain a z.s.s of length  $t$ , then  $|S| = t + d$  for  $d \in [6, 11]$ . Thus,  $\mathfrak{s}'_t(I_3) = t + 6$ .  $\square$

**Remark 15.** Aaron Berger [4] has recently announced a proof of Conjecture 6.

### 3. APPENDIX

In this section, we include Zhong's proofs of Lemma 3 and Remark 4.

*Proof of Lemma 3.*

(i) Since  $\mathfrak{s}(G) \leq \mathfrak{s}'(G)$ , it suffices to prove that  $\mathfrak{s}'(G) \geq \mathfrak{s}(G)$ . Let  $S = \prod_{i=1}^{\mathfrak{s}(G)-1} g_i$  be a sequence in  $\mathcal{F}(G)$  of length  $|S| = \mathfrak{s}(G) - 1$  such that  $S$  has no zero-sum subsequence of length  $\exp(G)$ . Assume that  $\sigma(S) = h \in G$  and let  $t \in \mathbb{N}$  be such that  $(\mathfrak{s}(G) - 1)t \equiv 1 \pmod{\exp(G)}$ . Then  $(\mathfrak{s}(G) - 1)th = h$  in  $G$ . Define  $S' = \prod_{i=1}^{\mathfrak{s}(G)-1} (g_i - th)$ . Since

$\sigma(S') = \sigma(S) - (s(G) - 1)th = 0$  and  $S'$  does not contain a zero-sum subsequence of length  $\exp(G)$ , it follows that  $s'(G) \geq s(G)$ .

(ii) Let  $S \in \mathcal{B}(G)$  be such that  $|S| = s(G) - 1$ . We want to prove that  $S$  contains a zero-sum subsequence of length  $n = \exp(G)$ . If we assume to the contrary that  $S$  does not contain a zero-sum subsequence of length  $n$ , then Property D implies that there exists  $T \in \mathcal{F}(G)$  such that  $S = T^{[n-1]}$ . Thus,  $|T| = c$  and  $\sigma(T) = 0$ . Therefore  $T^{[n/c]}$  is a zero-sum sequence of length  $n$ , a contradiction.  $\square$

*Proof of Remark 4.*

(i) Let  $n$  be odd and  $G \cong \mathbb{Z}_n^2$ . Since  $s(G) = 4n - 3$ , then  $\gcd(s(G) - 1, n) = 1$ . Thus,  $s(G) = s'(G)$  by Lemma 3(i).

(ii) Let  $h \geq 2$  be an integer and  $G \cong \mathbb{Z}_{2^h}^2$ . Then  $\exp(G) = 2^h$ ,  $s(G) = 4(2^h - 1) + 1$ ,  $\gcd(s(G) - 1, \exp(G)) = 4$ , and  $G$  has Property D (by [13, Theorem 3.2]). Thus, Lemma 3(ii) yields  $s'(G) < s(G)$ . Since  $\gcd(s(G) - 2, \exp(G)) = 1$ , the proof of Lemma 3(i) yields  $s'(G) > s(G) - 2$ . Thus,  $s'(G) = s(G) - 1$ .  $\square$

**Acknowledgement:** We thank Alfred Geroldinger for providing references and for his valuable comments which helped clarify the definitions and terminology. We also thank Qinghai Zhong for allowing us to include Lemma 3 and Remark 4.

## REFERENCES

- [1] P. Baginski, S. Chapman, R. Rodriguez, G. Schaeffer, and Y. She, On the Delta set and catenary degree of Krull monoids with infinite cyclic divisor class group, *J. Pure Appl. Algebra* **214** (2010), 1334–1339.
- [2] N. Baeth, A. Geroldinger, Monoids of modules and arithmetic of direct-sum decompositions *Pacific J. Math.*, **271** (2014), 257–319.
- [3] N. Baeth, A. Geroldinger, D. Gryniewicz, and D. Smertnig, A semigroup theoretical view of direct-sum decompositions and associated combinatorial problems, *J. Algebra and its Appl.*, **14** (2015), 60 pp.
- [4] A. Berger, The Maximum Length of  $k$ -bounded,  $t$ -avoiding Zero-sum Sequences over  $\mathbb{Z}$ , <http://arxiv.org/pdf/1608.04125>.
- [5] Y. Caro, Zero-sum problems – a survey, *Discrete Math.* **152** (1996), 93–113.
- [6] S. Chapman, W. Schmid, and W. Smith, On minimal distances in Krull monoids with infinite class group, *Bull. London Math. Soc.* **40**(4) (2008), 613–618.
- [7] P. Diaconis, R. Graham, and B. Sturmfels, Primitive partition identities, *Paul Erdős is 80, Vol. II, Janos Bolyai Society*, Budapest, (1995), 1–20.
- [8] P. Erdős, A. Ginzburg and A. Ziv, A theorem in additive number theory, *Bull. Res. Council Israel* **10F** (1961), 41–43.
- [9] Y. Fan, W. Gao, Q. Zhong, On the Erdős-Ginzburg-Ziv constant of finite abelian groups of high rank, *J. Number Theory* **131** (2011), 1864–1874.
- [10] W. Gao, Zero sums in finite cyclic groups, *Integers* **0**, #A12 (2000) (electronic).
- [11] W. Gao and A. Geroldinger, Zero-sum problems in finite abelian groups: A survey, *Expo. Math.* **24** (2006), 337–369.

- [12] A. Geroldinger and F. Halter-Koch, Non-unique factorizations: a survey, in *Multiplicative ideal theory in commutative algebra*, Springer, New York, (2006), 207–226.
- [13] W. Gao, A. Geroldinger, and W. Schmid, Inverse zero-sum problems, *Acta Arith.* **128** (2007), 245–279.
- [14] W. Gao, D. Han, J. Peng, and F. Sun, On zero-sum subsequences of length  $k \exp(G)$ , *J. Combin. Theory Ser. A* **125** (2014), 240–253.
- [15] A. Geroldinger and Q. Zhong, *Personal Communication*.
- [16] J. Lambert, Une borne pour les générateurs des solutions entières positives d’une équation diophantienne linéaire, *C. R. Acad. Sci. Paris Ser. I Math.* **305** (1987), 39–40.
- [17] A. Plagne and S. Tringali, The Davenport constant of a box, *Acta Arith.* **171.3** (2015), 197–220.
- [18] D. Gryniewicz, *Structural Additive Theory*, Springer International, Switzerland, 2013.
- [19] C. Reiher, On Kemnitz’ conjecture concerning lattice-points in the plane, *Ramanujan J.* **13**, 333–337.
- [20] M. Sahs, P. Sissokho, and J. Torf, A zero-sum theorem over  $\mathbb{Z}$ , *Integers* **13** (2013), #A70.
- [21] P. Sissokho, A note on minimal zero-sum sequences over  $\mathbb{Z}$ , *Acta Arith.* **166** (2014), 279–288.
- [22] Q. Zhong, *Personal Communication*.